Mechanism Design with Correlated Information

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Workshop on Mechanism Design

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Optimal mechanism design

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When bidder information is correlated

- There exists a mechanism that is efficient and leaves bidders with zero expected surplus
- Full-surplus extraction

Symmetric Model with Correlated Information

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- All this is common knowledge

Model (continued)

- From P define a matrix of conditional probabilities, P_1
- P_1 is a $m \times m$ matrix with elements $\Pr(x_2|x_1)$, $x_1, x_2 \in \mathsf{X}$
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- **Assumption:** P_1 and P_2 are full-rank matrices
- **Assumption:** Single-crossing condition is satisfied

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At $X_1 = x_1$, for any $x_1 \in \mathsf{X}$, buyer 1's expected surplus is

$$u_1(x_1) - \sum_{x_2 \in \mathsf{X}} \Pr(x_2|x_1)c_1(x_2) \;=\; u_1(x_1) - u_1(x_1) \;=\; 0$$

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$$P_i = \begin{bmatrix} \Pr(0 \mid 0) & \Pr(0 \mid 1) \\ \Pr(1 \mid 0) & \Pr(1 \mid 1) \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

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Efficient allocation rule: $Q(X_1, X_2) =$ (prob 1 gets it, prob 2 gets it)

$$Q(X_1, X_2) \;=\; \begin{bmatrix} (0,0) & (0,1) \\ (1,0) & (0.5,0.5) \end{bmatrix}$$

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$$\mathbf{u_i} - P_i \mathbf{c_i} = \begin{pmatrix} 0 - [(0.75)(-0.125) + (0.25)(0.375)] \\ 0.25 - [(0.25)(-0.125) + (0.75)(0.375)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Caveats

The optimal mechanism is

- Is not detail free
- Not ex post individually rational
- Does not work with limited liability